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Abstract. A class of quantum many-body models of arbitrary dimension and arbitrary statistics of particles, for which exact eigenstates may be obtained, is found. It is assumed that: (i) models contain two (or $2m$) kinds of particles with 'symmetric' matrix elements of pairwise interaction (all potentials coincide with each other to within a sign and wavefunctions of free particles of two components coincide to within a phase factor; pairwise interactions are otherwise arbitrary); (ii) there exists the degeneracy of (the sum) of free-particle spectra. Exact many-body eigenstates correspond to a condensation of non-interacting composite particles ('excitons') which are not exactly bosons, into a single quantum state, and to excitations over the condensate. The origin of the possibility of exact solution is the symmetry under the continuous rotations in the isospin space of two components, to which Bogolubov canonical transformations with parameters u, v independent of momentum correspond. The class of such models comprises, in particular, two-dimensional electron–hole systems in a strong magnetic field.

1. Introduction

There are a few known quantum many-body models which are exactly solvable. Most of these models are either one dimensional (Mattis and Lieb 1965, Lieb and Mattis 1966) or consider a situation with short-range pairwise interactions (Wada *et al* 1958, Anderson 1958; see also Thouless 1972, Gaudin 1983 and references therein). We intend to demonstrate that there exists a class of models of arbitrary dimension which allows one to find some exact many-body eigenstates for potentials of interaction of quite an arbitrary form, including long-range potentials.

The essential features of these models are (i) the presence in the system of two (or $2m$) kind of particles with 'symmetric' matrix elements of interaction, and (ii) the degeneracy of free-particle spectra (more precisely, the sum of the spectra is to be made degenerate).

Such models describe, e.g., 2D electrons (e) and holes (h) in magnetic field H, which is strong enough so that virtual transitions of particles between different Landau levels are negligible. This strong magnetic field approximation is valid when $E_0 \ll \omega_c$, where E_0 is the interaction energy per particle (for pure Coulomb interactions $E_0 \propto e^2/\kappa r_H \propto H^{1/2}$, $r_H = (c/eH)^{1/2}$ is the magnetic length), $\omega_c = eH/mc \propto H$ is the energy

separation between adjacent Landau levels; we set $\hbar = 1$. In this situation particles are confined to their uppermost (in the simplest case—zero) partially occupied macroscopically degenerated Landau levels.

Many-body effects in the ground state of this system have been analysed by means of temperature diagram technique (with successive exploration of the limit $T \to 0$) by Lerner and Lozovik (1981, 1982). It should be noted that usual perturbative methods at $T = 0$ are inapplicable because of the infinite degeneracy of the non-interacting ground state.

It has been shown diagrammatically that the ground state energy of a 2D e-h system in a strong magnetic field can be obtained exactly. It turned out to be equal to the additive sum of the binding energies of 2D magnetoexcitons of zero momentum. Direct quantum mechanical consideration provided the ground-state wavefunction, which does have the form of the condensate of 2D magnetoexcitons; the property of the ground state as an ideal gas of excitons has been reaffirmed by non-perturbative means and some excited states of the system have been found (Dzyubenko and Lozovik 1983a, b). The wavefunction of the ground state has also been considered by Bychkov *et al* (1983); in that work, however, connection to the condensation of excitons has not been demonstrated explicitly.

Other closely related exactly solvable models have also been found: the 2D layered (multicomponent) e-h system in strong field H, which describes a set of spatially separated quantum wells, each containing 2D electrons and holes; the 2D e-h system in crossed electric and strong magnetic fields (Dzyubenko and Lozovik 1984, 1986).

The interesting aspect of the situation in crossed fields is the possibility of non-dissipative energy transport by excitons in the non-equilibrium system. This effect may be considered as the analogue of the quantum Hall effect (for a review see, e.g., Prange and Girvin 1987) for the case of a neutral 2D two-component system (Dzyubenko and Lozovik 1984). Very close consideration was later given by Paquet *et al* (1985); see also Rice *et al* (1985).

2D electron systems with equivalent groups of carriers in strong magnetic fields turn out to have the same symmetry. The excitations of the 'excitonic' kind in this case are 'valley-waves' in 2D multivalley semiconductors in strong fields H (Rasolt *et al* 1986), and, when the spin of electrons is taken into account, are $k = 0$ spin-wave excitations in which electrons are excited to a higher Landau level with the same number but the opposite direction of spin (the dispersion relations for such excitations have been considered by Bychkov *et al* 1981 and Kallin and Halperin 1984).

The aim of this paper is to point out the essential features of the class of such exactly solvable many-body models. In section 2 we derive a many-body Hamiltonian with the emphasis on the formal requirements on wavefunctions and interaction potentials. In section 3 we present a very simple consideration based on the operator algebra for quantum equations of motion. In section 4 the general consideration is illustrated by the example of a 2D e-h system in a strong magnetic field. Section 5 is devoted to the detailed analysis of the exact continuous symmetry of Hamiltonians and, finally, in section 6 we study the arbitrary statistics of particles.

2. Hamiltonians

We consider a many-body system consisting of two kinds of particles. The Hamiltonian

of free particles has the form

$$H_0 = \sum_{i=1,2} \sum_p \varepsilon_i(p) a_{ip}^{\dagger} a_{ip} \tag{1}$$

where the index $i = 1, 2$ distinguishes between the two kinds of particles and the fermion operator $a_{ip}^{\dagger}$ creates the particle of the kind i in the state with (quasi)continuous quantum number p. It may be considered as a d-component (quasi)momentum (d does not necessarily coincide with the dimensionality of real space D); p may take N_0 different values (for 2D electrons and holes in a strong magnetic field, $N_0 = L^2/2\pi r_H^2$ is the macroscopic degeneracy of Landau levels, L^2 is the area of the system).

We consider the case when the dispersion relations of the components satisfy the condition

$$\varepsilon_1(p) + \varepsilon_2(-p) = \varepsilon_0 \tag{2}$$

where ε_0 is a constant independent of p (for a semiconductor ε_0 is the gap energy).

Note, that we use *the hole description* of one kind of particle (say, with the number 2). When the electron description is used, it corresponds to

$$\varepsilon_2(p) \to -\varepsilon_2(-p) \qquad a_{2p}^{\dagger} \to a_{2,-p}$$

and (see below)

$$U_{12}(r) \to -U_{12}(r) \qquad \phi_{2p}(r) \to \phi_{2,-p}^*(r).$$

The Hamiltonian of interaction is given by

$$H_{\text{int}} = \frac{1}{2} \sum_{i,j=1,2} \sum_{p_1,\ldots,p_2'} U_{ij}(p_1, p_2; p_1', p_2') a_{ip_1}^{\dagger} a_{jp_2}^{\dagger} a_{jp_2'} a_{ip_1'}. \tag{3}$$

Interaction matrix elements must have the following properties:

$$U_{ij}(p_1, p_2; p_1', p_2') = U_{ji}(p_2, p_1; p_2', p_1') \tag{4}$$

$$U_{11}(p_1, p_2; p_1', p_2') = -U_{12}(p_1, -p_2'; p_1', -p_2)$$

$$= U_{22}(-p_2', -p_1'; -p_2, -p_1) \tag{5}$$

$$U_{11}(p_1, p_2; p_1', p_2') = \delta_{p_1+p_2, p_1'+p_2'} V(p_1, p_1', p_2') \tag{6}$$

$$V(p_1, p_1', p_2') = v(p_1 - p_1', p_1 - p_2'). \tag{7}$$

To clarify the conditions (4)–(7), we shall make use of the representation in real space. Let $\phi_{ip}(r)$ be the wavefunctions of particles in the states corresponding to $a_{ip}^{\dagger}$, interacting via potentials $U_{ij}(r)$ (we set $U_{12}(r) \equiv U_{21}(r)$), and interaction matrix elements are usually defined as

$$U_{ij}(p_1, p_2; p_1', p_2') = \int dr_1 \int dr_2 \, \phi_{ip_1}^*(r_1) \phi_{jp_2}^*(r_2) U_{ij}(r_1 - r_2) \phi_{jp_2'}(r_2) \phi_{ip_1'}(r_1). \tag{8}$$

Then (4) follows from $U_{ij}(r) = U_{ij}(-r)$, which is the general property of pairwise potentials, and (5) follows from

$$U_{11}(r) = U_{22}(r) = -U_{12}(r) = U(r) \tag{9}$$

$$\phi_{1p}^*(r)\phi_{1p'}(r) = \phi_{2,-p}(r)\phi_{2,-p'}^*(r) \tag{10}$$

i.e. interactions between particles of the same kind coincide with each other and (with the reversed sign) with that of different kinds; besides, the wavefunctions of particles

of the two kinds in the *electron* representation (see above) coincide with each other (to within a phase factor). Equations (6) and (7) mean that momentum is conserved and that the matrix element $V(p_1, p_1', p_2')$ depends only on the differences of its arguments.

It is worth noting here that the above requirements on *interactions* in many-body system are not the exotic ones. In fact, conditions (9) and (10) are fulfilled for, e.g., neutral Coulomb systems with particles (with charges of different signs) which are described by the plane waves. Also, when $\phi_{jp}(r)$ are plane waves, $v(p_1 - p_1', p_1 - p_2')$ is, as usual, the Fourier transform of the interaction potential $\tilde{U}(p_1 - p_1')$.

3. Quantum equation of motion: finite algebra of operators

When conditions (2), (4)–(7) are satisfied, the Hamiltonian of the system $\hat{H} = H_0 + H_{int}$ allows one to find some exact many-body eigenstates. To demonstrate this, let us introduce the creation operator of the composite particle ('exciton')

$$Q_0^\dagger = \frac{1}{\sqrt{N_0}} \sum_p a_{2p}^\dagger a_{1,-p}^\dagger \tag{11}$$

with zero momentum of the centre of mass $p_1 + p_2 = 0$. As we shall see below, $Q_0^\dagger$ describes the pair of particles the most tightly bound in r space (yet compatible with the uncertainty principle).

The exact quantum equation of motion for $Q_0^\dagger$ quite unexpectedly has the form of the finite algebra of operators:

$$[\hat{H}, Q_0^\dagger] = \varepsilon Q_0^\dagger \tag{12}$$

where

$$\varepsilon = \varepsilon_0 + E_0 \qquad E_0 = -\sum_p v(p, 0) \tag{13}$$

and ε_0, $v(p, p')$ are defined by (2) and (7), respectively (for a straightforward, but rather tedious, derivation see the appendix).

From equation (12) it follows at once that

$$\hat{H}(Q_0^\dagger)^N |0\rangle = N\varepsilon (Q_0^\dagger)^N |0\rangle \tag{14}$$

where $|0\rangle$ denotes the vacuum state. Hence the state with the condensate of composite particles $(Q_0^\dagger)^N |0\rangle$ is the exact eigenstate of the many-body Hamiltonian.

The question as to whether this state is the ground state of the many-body system, in the absence of general theorems, may only be solved by exploring a concrete physical situation. For application to the 2D electron–hole system in a strong magnetic field see Lerner and Lozovik (1981); see also Paquet *et al* (1985) and references therein.

From (12) it also follows that particles $Q_0^\dagger$ do correspond to the ideal gas in the sense that they do not interact either with each other or with other particles. Indeed, let $f^\dagger$ be the creation operator of a overcondensate excitation consisting only of creation operators $a_{ip}^\dagger$, such that $f^\dagger |0\rangle$ is the eigenvector of Hamiltonian $\hat{H}$ with the eigenvalue E_f. Then, taking into account the commutation relation $[f^\dagger, Q_0^\dagger] = 0$, we have

$$\hat{H} f^\dagger (Q_0^\dagger)^N |0\rangle = (N\varepsilon + E_f) f^\dagger (Q_0^\dagger)^N |0\rangle. \tag{15}$$

The same holds true for the absence of interaction of particles $Q_0^\dagger$ with the external fields such that

$$V_1(r) = -V_2(r) \equiv V(r). \tag{16}$$

Indeed, the interactions with the field V are described by the Hamiltonian

$$\hat{V} = \sum_{i=1,2} \sum_p V_i(p', p) a_{ip'}^\dagger a_{ip}. \tag{17}$$

From (10) and (16) it follows that $V_1(p', p) = -V_2(-p, -p')$ and hence (Dzyubenko 1989, 1990)

$$[\hat{V}, Q_0^\dagger] = 0. \tag{18}$$

4. 2D electrons and holes on the lowest Landau levels

In this section we shall illustrate the above consideration by the example of 2D electrons and holes in a strong magnetic field. Examples of other related exactly solvable models will be given elsewhere.

In the Landau gauge of the vector potential $A = (0, Hx, 0)$, the wavefunctions of 2D electrons and holes in their lowest $n = 0$ Landau levels are given by

$$\phi_{ip_y}(r) = \frac{1}{(L\sqrt{\pi} r_H)^{1/2}} \exp(ip_y y - (x \pm p_y r_H^2)^2 / 2r_H^2) \tag{19}$$

where the one-dimensional momentum p_y determines the x coordinate of the centre of a cyclotron orbit of electrons and holes X by the relations, respectively $X = \mp p_y r_H^2$.

Hence the creation operator of a 2D magnetoexciton of zero momentum $Q_0^\dagger$ (11) describes the most tightly bound e-h pair with zero mean interparticle separation $\langle r_{eh} \rangle = 0$ (see also Lerner and Lozovik (1980) and references therein). Since the density probabilities for electrons and holes are exactly equal (see (10)), composite particles $Q_0^\dagger$ can be regarded as completely neutral objects. This gives some intuitive explanation of the ideal character of such particles.

From (13) and (19) we obtain the binding energy of a 2D magnetoexciton of zero momentum

$$E_0 = -\int \frac{d^2 q}{(2\pi)^2} \tilde{U}(q) \exp(-q^2 r_H^2 / 2) \tag{20}$$

where $\tilde{U}(q)$ is the Fourier transform of interaction potential $U(r)$ (9). For the Coulomb interaction $U(r) = e^2/\kappa r$, equation (20) gives $E_0 = -(\pi/2)^{1/2} e^2/\kappa r_H$.

From the eigenvalue equation (15) it follows, for example, that the following problems turn out to be exactly solvable.

(i) One excess carrier over the condensate of excitons:

$$f^\dagger = a_{ip}^\dagger \qquad E_f = \varepsilon_i(p)$$

which means that the electron (or hole) does not polarize the condensate of excitons.

(ii) The e-h pair which forms the exciton with 2D magnetic momentum P over the condensate:

$$f^\dagger = Q_P^\dagger = \frac{1}{\sqrt{N_0}} \sum_{p_y} \exp(iP_x p_y r_H^2) a_{2,P_y/2+p_y}^\dagger a_{1,P_y/2-p_y}^\dagger \tag{21}$$

$$E_f = E(P) = -\int \frac{d^2 q}{(2\pi)^2} \tilde{U}(q) \exp(iqP r_H^2 - q^2 r_H^2 / 2) \tag{22}$$

hence the spectrum of two-particle excitations can be obtained exactly for this system and is exactly determined by the magnetoexciton dispersion relation (22) (Lerner and Lozovik 1981, Dzyubenko and Lozovik 1983a, b).

(iii) Two or three excess carriers with charges of the same sign which, in spite of repulsive interaction, form in a strong field H bound states with discrete spectra (Bychkov *et al* 1981, Laughlin 1983).

The concluding remark of this section is concerned with equation (18). For 2D magnetoexcitons of zero momentum it implies that the interactions with external fields of the form (16) are absent, unless virtual transitions to higher Landau levels are taken into account. In the strong magnetic field limit this can be done perturbatively, or—just as in the case of the uniform electric field $\mathscr{E}$—by the exact inclusion of the external field in the zero order (Dzyubenko and Lozovik 1984).

The important exception is the situation when the rearrangement of the exciton ground state in the external field occurs. This takes place for 2D magnetoexcitons in the presence of, e.g., a Coulomb impurity, where the formation of impurity-bound states becomes energetically favourable (Dzyubenko 1989, 1990). This effect, obviously, cannot be treated perturbatively starting from the delocalized magnetoexciton $Q_0^\dagger$.

5. The symmetry of the Hamiltonian

The origin of the finite operator algebra (12) lies in the exact isospin symmetry of the two components

$$a_{1,p} \rightleftarrows a_{2,-p}^\dagger.$$

It was explicitly used for obtaining the ground state of a 2D e–h system on the lowest Landau levels by Dzyubenko and Lozovik (1983a, b) (compare Rice *et al* 1985). A system with equivalent groups of 2D electrons in a multivalley semiconductor in strong field H possesses, as was established by Rasolt *et al* (1986), the same symmetry.

The existence of this symmetry can be demonstrated as follows. Consider Bogolubov's canonical transformations, which mix the two components, with real transformation parameters u, v independent of quantum number p. In the case of Fermi statistics the transformations have the form (Bogolubov 1958)

$$a_{1,p} \rightarrow \tilde{a}_{1,p} = u a_{1,p} + v a_{2,-p}^\dagger \qquad a_{2,p} \rightarrow \tilde{a}_{2,p} = u a_{2,p} - v a_{1,-p}^\dagger \qquad (23)$$

where $u^2 + v^2 = 1$. As is known, transformations (23) may be presented as a result of the rotation in the isospin space of two components. In the case considered canonical transformations (23) are generated by the unitary operator

$$S = \exp[\Theta(Q_0^\dagger - Q_0)] \qquad (24)$$

so that $\tilde{a}_{ip} = S a_{ip} S^\dagger$ and $u = \cos(\Theta/\sqrt{N_0})$, $v = \sin(\Theta/\sqrt{N_0})$.

It should be stressed that in our case the anti-Hermitian generator of rotations $\hat{L} = Q_0^\dagger - Q_0$ turns out to be directly connected with the creation operator of composite particle $Q_0^\dagger$. It is this fact, together with equation (28) below, that points to the possibility of exact solution.

Since transformations (23) do not conserve numbers of particles, it is convenient to consider the Hamiltonian

$$\tilde{H} = \hat{H} - \mu \hat{N} \qquad \hat{N} = \hat{N}_1 + \hat{N}_2 = \sum_i \sum_p a_{ip}^\dagger a_{ip} \qquad . \qquad (25)$$

where $\hat{N}_i$ are the particle number operators, μ being equal chemical potentials of components.

The Hamiltonian transformed $S\tilde{H}S^\dagger$ becomes

$$S\tilde{H}S^\dagger = W(\Theta) + \bar{H}_0 + H_{int} \tag{26}$$

where $W = (\varepsilon - 2\mu)v^2 N_0$ is the numerical function of Θ (c-number),

$$\bar{H}_0 = -uv(\varepsilon - 2\mu) \sum_p (a_{2p}^\dagger a_{1,-p}^\dagger + \text{HC}) - [v^2(\varepsilon - 2\mu) + \mu]\hat{N} \tag{27}$$

is the bilinear part of the Hamiltonian, and H_{int} is the interaction Hamiltonian, which exactly conserved its initial form (3).

It turns out that letting $\mu = \mu_0 = \varepsilon/2$, which corresponds to 'Bose' condensation of composite particles into the state with the energy $2\mu_0 = \varepsilon$, we obtain the Hamiltonian $\tilde{H}$ which is exactly invariant under the rotations

$$S\tilde{H}S^\dagger = \tilde{H}. \tag{28}$$

This quite unexpected exact symmetry formally follows from two facts. Firstly, the Bogolubov transformations, with parameters u, v independent of momentum, do not generate out of the interaction Hamiltonian of the two-component 'symmetric' system (in the sense of (9) and (10)) non-diagonal terms of the form

$$a_1^\dagger a_2^\dagger a_1^\dagger a_2^\dagger \qquad a_1^\dagger a_2^\dagger a_1^\dagger a_1 \qquad a_2^\dagger a_1^\dagger a_2^\dagger a_1$$

and their Hermitian conjugates (compare, e.g., Keldysh and Kozlov 1968). Secondly, with the condition (2), the only non-diagonal terms in the Hamiltonian (27), namely, $a_{2p}^\dagger a_{1,-p}^\dagger$, are multiplied by a *constant* (rather than a function of momentum p), which can be set to zero by the appropriate choice of chemical potential μ.

From (28) it follows that the generator $\hat{L}$ is the integral of motion, i.e. $[\hat{L}, \tilde{H}] = 0$. Taking into account that $\tilde{H}$ is the Hermitian operator, one obtains $[\tilde{H}, Q_0^\dagger] = 0$, thus

$$[\hat{H}, Q_0^\dagger] = \mu_0[\hat{N}, Q_0^\dagger] = \varepsilon Q_0^\dagger \tag{29}$$

which strictly coincides with the equation of motion (12).

From (28) it also follows that

$$\hat{H}S^\dagger|0\rangle = \mu_0 \hat{N}S^\dagger|0\rangle \tag{30}$$

where

$$S^\dagger|0\rangle = \prod_p (u - va_{2p}^\dagger a_{1,-p}^\dagger)|0\rangle \tag{31}$$

is the BCS-like state (Bardeen *et al* 1957), which does not have definite particle number and describes in this case the coherent state of excitons. Hence, for the systems under consideration the BCS like state (31) is the exact many-body state (the eigenstate of the Hamiltonian $\tilde{H} = H - \mu_0\hat{N}$ with the eigenvalue which is equal to zero).

Acting on both sides of equation (30) by the operator $\hat{P}_{N,N}$, projecting onto the states with $N_1 = N_2 = N$ particles, with the use of commutation relations

$$[\hat{H}, \hat{P}_{N,N}] = [\hat{N}, \hat{P}_{N,N}] = 0 \tag{32}$$

which follow from the fact that both operators $\hat{H}$, $\hat{N}$ conserve particle numbers, we obtain

$$\hat{H}(\hat{P}_{N,N}S^\dagger)|0\rangle = \varepsilon N(\hat{P}_{N,N}S^\dagger)|0\rangle \tag{33}$$

$$(\hat{P}_{N,N}S^\dagger)|0\rangle = \text{constant} \times (Q_0^\dagger)^N|0\rangle. \tag{34}$$

Thus the continuous symmetry under the rotations in the isospin space of the components actually exists.

6. Arbitrary statistics of components

It turns out that the quantum equations of motion (12) are unchanged, as one may verify, when the statistics of one (or both) components is changed from Fermi to Bose statistics. This may be considered as the additional discrete symmetry of the system (Dzyubenko (1986); see also Dzyubenko and Lozovik (1989)).

Therefore the composite particles $Q_0^\dagger$ correspond to the ideal gas irrespective of the statistics of the components. Hence the eigenvalue equation (14) holds (at least, formally) for arbitrary statistics.

It should be noted, however, that

(i) the possible number of particles N in the state $(Q_0^\dagger)^N|0\rangle$ is connected with the statistics of the components;

(ii) it becomes possible to condense into a single quantum state for composite particles $Q_0^\dagger$ which are not exactly bosons. The commutation relations for operators $Q_0^\dagger$ are of the form

$$[Q_0^\dagger, Q_0^\dagger] = 0 \qquad [Q_0, Q_0^\dagger] = 1 \mp \frac{\hat{N}_1 + \hat{N}_2}{N_0} \qquad \begin{array}{l} a_{ip} \text{ fermions} \quad (35) \\ a_{ip} \text{ bosons} \quad (36) \end{array}$$

$$[Q_0^\dagger, Q_0^\dagger]_+ = 0 \qquad [Q_0, Q_0^\dagger]_+ = 1 - \frac{\hat{N}_1 - \hat{N}_2}{N_0} \qquad \begin{array}{l} a_{1p} \text{ fermions} \\ a_{2p} \text{ bosons} \end{array} \quad (37)$$

where $\hat{N}_i$ are the particle number operators and $[\ ,\ ]_+$ denotes the anticommutator.

It follows from the first of equations (37) that composite particles $Q_0^\dagger$ with half-integer total spin obey, as is well known, the Pauli exclusion principle: possible filling numbers for them are $N = 0, 1$. It should be noted, however, that contrary to a widespread opinion, due to the operator term on the right-hand side of (37), $Q_0^\dagger$ are not, strictly speaking, fermions.

Composite particles with integer total spin (the cases (35) and (36)) may be considered as bosons, as was first pointed out for 3D excitons by Keldysh and Kozlov (1968), only in the limit of small densities $N_i \ll N_0$. When both kinds of 'internal' particles are fermions (35), in the case when $N_0 - N \ll N_0$ the right-hand side of $(35) \simeq -1$, and *anti-excitons* are nearly bosons; it corresponds to the interchange $Q_0^\dagger \rightleftarrows Q_0$.

In the case (35), the restriction on the possible number of composite particles $Q_0^\dagger$ in the condensate N follows from the Pauli exclusion principle, i.e. $N \le N_0$. In the case of Bose statistics of components (36), N may obviously be arbitrary.

The last two statements also follow from the explicit expressions for the state with the condensate

$$\langle 0|(Q_0)^N(Q_0^\dagger)^N|0\rangle = \begin{cases} N!\left[\dfrac{N_0!}{N_0^N(N_0 - N)!}\right] & N \le N_0 \\ 0 & N > N_0 \end{cases} \qquad a_{ip} \text{ fermions} \quad (38)$$

$$\langle 0|(Q_0)^N(Q_0^\dagger)^N|0\rangle = N!\left[\frac{(N_0 + N - 1)!}{N_0^N(N_0 - 1)!}\right] \qquad a_{ip} \text{ bosons.} \quad (39)$$

Note that the factors in the square brackets of equations (38) and (39) are due to the deviation of statistics of particles $Q_0^\dagger$ from pure bose statistics.

In conclusion, the class of 'symmetric' two-component exactly solvable quantum models has been found. Exact many-body states correspond to the condensation of

non-interacting two-body composite particles and excitations over the condensate. The symmetry between the components implies, actually, that there are no multi-particle correlations in the state with the condensate. This can be regarded as the basis for the possibility of an exact solution.

For the two-component models with close but not exactly 'symmetric' properties, our consideration may be useful as a good 'zero-order' approximation. Models of such a kind are 2D electron–hole systems (in semiconductor quantum wells) and multi-component 2D electron systems in a strong magnetic field, where the kinetic energy of particles is 'quenched'. Other physical realizations, maybe among discrete spin systems, are also possible.

Acknowledgment

Appendix

For the Hamiltonian of free particles H_0 (1), using equation (2), one easily obtains

$$[H_0, Q_0^\dagger] = \frac{1}{\sqrt{N_0}} \sum_p (\varepsilon_1(-p) + \varepsilon_2(p)) a_{2p}^\dagger a_{1,-p}^\dagger = \varepsilon_0 Q_0^\dagger. \tag{A.1}$$

Consider now the three parts of the interaction Hamiltonian (3), $H_{\text{int}} \equiv H_{\text{int}}^{(11)} + H_{\text{int}}^{(22)} + H_{\text{int}}^{(12)}$, separately; here, e.g., $H_{\text{int}}^{(12)}$ describes the interactions of particles of different kinds ($i \neq j$ in (3)).

For interactions of particles of the same kinds $H_{\text{int}}^{(11)}$ and $H_{\text{int}}^{(22)}$, after the redefinition of indices of summation $p_1 \rightleftarrows p_2$ and $p_1' \rightleftarrows p_2'$, with the use of equation (4), we obtain, respectively,

$$[H_{\text{int}}^{(11)}, Q_0^\dagger] = -\frac{1}{\sqrt{N_0}} \sum_{p_1, \dots} U_{11}(p_1, p_2; p_1', p_2') a_{1p_1}^\dagger a_{1p_2}^\dagger a_{2,-p_2'}^\dagger a_{1p_1'} \tag{A.2}$$

$$[H_{\text{int}}^{(22)}, Q_0^\dagger] = \frac{1}{\sqrt{N_0}} \sum_{p_1, \dots} U_{22}(p_1, p_2; p_1', p_2') a_{2p_1}^\dagger a_{2p_2}^\dagger a_{1,-p_2'}^\dagger a_{2p_1'}. \tag{A.3}$$

For $H_{\text{int}}^{(12)}$ we have

$$[H_{\text{int}}^{(12)}, Q_0^\dagger] = \frac{1}{\sqrt{N_0}} \sum_{p_1, \dots} U_{12}(p_1, p_2; p_1', p_2')$$
$$\times (-\delta_{p_1', -p_2'} a_{1p_1}^\dagger a_{2p_2}^\dagger + a_{1p_1}^\dagger a_{2p_2}^\dagger a_{2,-p_1'}^\dagger a_{2p_2'} + a_{1p_1}^\dagger a_{2p_2}^\dagger a_{1,-p_2'}^\dagger a_{1p_1'}). \tag{A.4}$$

Consider the first term in brackets in (A.4). Taking into account equations (5)-(7), we have

$$U_{12}(p_1, p_2; p_1', p_2') = -\delta_{p_1 + p_2, p_1' + p_2'} v(p_1 - p_1', p_1 + p_2). \tag{A.5}$$

Hence, passing in summation to variables $p = p_1 - p_1'$ and $q = -p_1$, we obtain

$$\frac{1}{\sqrt{N_0}} \sum_{p_1, \dots} \delta_{p_1, -p_2} \delta_{p_1', -p_2'} v(p_1 - p_1', p_1 + p_2) a_{1,-p_2}^\dagger a_{2p_2}^\dagger$$
$$= \left(-\sum_p v(p, 0)\right) \frac{1}{\sqrt{N_0}} \sum_q a_{2q}^\dagger a_{1,-q}^\dagger \equiv E_0 Q_0^\dagger. \tag{A.6}$$

Each of the remaining two terms in (A.4) are cancelled exactly by (A.3) and (A.2), respectively. Indeed, consider, e.g., the second term in the brackets of (A.4). By the following redefinition of the indices of summation:

$$p_1 \to -p_2' \qquad p_2 \to p_1 \qquad -p_1' \to p_2 \qquad p_2' \to p_1' \qquad (A.7)$$

and using (5), we obtain

$$\frac{1}{\sqrt{N_0}} \sum_{p_1,\cdots} U_{12}(-p_2', p_1; -p_2, p_1') a^\dagger_{1,-p_2'} a^\dagger_{2p_1} a^\dagger_{2p_2} a_{2p_1'}$$

$$= -\frac{1}{\sqrt{N_0}} \sum_{p_1,\cdots} U_{22}(p_2, p_1; p_2', p_1') a^\dagger_{2p_1} a^\dagger_{2p_2} a^\dagger_{1,-p_2'} a_{2p_1'}. \qquad (A.8)$$

From equation (4) it can be easily seen that (A.8) and (A.3) do cancel each other exactly. Hence, we have $[H_{\text{int}}, Q_0^\dagger] = E_0 Q_0^\dagger$ and, together with (A.1), the equation of motion (12).

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